

$$L = I\omega = 1.67 \text{ kg} \cdot \text{m}^2(0.1\pi \text{ rad/s}) = 0.17\pi \text{ kg} \cdot \text{m}^2/\text{s}.$$

The angular momentum vector is directed out of the page in the $\hat{\mathbf{k}}$ direction since the robot arm is rotating counterclockwise.

- b. We must include the Mars rock in the calculation of the moment of inertia, so we have

$$I_{\text{Total}} = I_{\text{R}} + I_{\text{F}} + I_{\text{MR}} = 3.17 \text{ kg} \cdot \text{m}^2$$

and

$$L = I\omega = 3.17 \text{ kg} \cdot \text{m}^2(0.1\pi \text{ rad/s}) = 0.32\pi \text{ kg} \cdot \text{m}^2/\text{s}.$$

Now the angular momentum vector is directed into the page in the $-\hat{\mathbf{k}}$ direction, by the right-hand rule, since the robot arm is now rotating clockwise.

- c. We find the torque when the arm does not have the rock by taking the derivative of the angular momentum using **Equation 11.8** $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$. But since $L = I\omega$, and understanding that the direction of the angular momentum and torque vectors are along the axis of rotation, we can suppress the vector notation and find

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha = \sum \tau,$$

which is Newton's second law for rotation. Since $\alpha = \frac{0.1\pi \text{ rad/s}}{0.1 \text{ s}} = \pi \text{ rad/s}^2$, we can calculate the net torque:

$$\sum \tau = I\alpha = 1.67 \text{ kg} \cdot \text{m}^2(\pi \text{ rad/s}^2) = 1.67\pi \text{ N} \cdot \text{m}.$$

Significance

The angular momentum in (a) is less than that of (b) due to the fact that the moment of inertia in (b) is greater than (a), while the angular velocity is the same.



11.3 Check Your Understanding Which has greater angular momentum: a solid sphere of mass m rotating at a constant angular frequency ω_0 about the z -axis, or a solid cylinder of same mass and rotation rate about the z -axis?



Visit the **University of Colorado's Interactive Simulation of Angular Momentum** (<https://openstaxcollege.org//21angmomintsim>) to learn more about angular momentum.

11.3 | Conservation of Angular Momentum

Learning Objectives

By the end of this section, you will be able to:

- Apply conservation of angular momentum to determine the angular velocity of a rotating system in which the moment of inertia is changing
- Explain how the rotational kinetic energy changes when a system undergoes changes in both moment of inertia and angular velocity

So far, we have looked at the angular momentum of systems consisting of point particles and rigid bodies. We have also analyzed the torques involved, using the expression that relates the external net torque to the change in angular momentum, **Equation 11.8**. Examples of systems that obey this equation include a freely spinning bicycle tire that slows over time due to torque arising from friction, or the slowing of Earth's rotation over millions of years due to frictional forces exerted on tidal deformations.

However, suppose there is no net external torque on the system, $\sum \vec{\tau} = 0$. In this case, **Equation 11.8** becomes the **law of conservation of angular momentum**.

Law of Conservation of Angular Momentum

The angular momentum of a system of particles around a point in a fixed inertial reference frame is conserved if there is no net external torque around that point:

$$\frac{d\vec{L}}{dt} = 0 \quad (11.10)$$

or

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \text{constant}. \quad (11.11)$$

Note that the *total* angular momentum $\vec{L}$ is conserved. Any of the individual angular momenta can change as long as their sum remains constant. This law is analogous to linear momentum being conserved when the external force on a system is zero.

As an example of conservation of angular momentum, **Figure 11.14** shows an ice skater executing a spin. The net torque on her is very close to zero because there is relatively little friction between her skates and the ice. Also, the friction is exerted very close to the pivot point. Both $|\vec{F}|$ and $|\vec{r}|$ are small, so $|\vec{\tau}|$ is negligible. Consequently, she can spin for quite some time. She can also increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

$$L' = L$$

or

$$I' \omega' = I \omega,$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because I' is smaller, the angular velocity ω' must increase to keep the angular momentum constant.

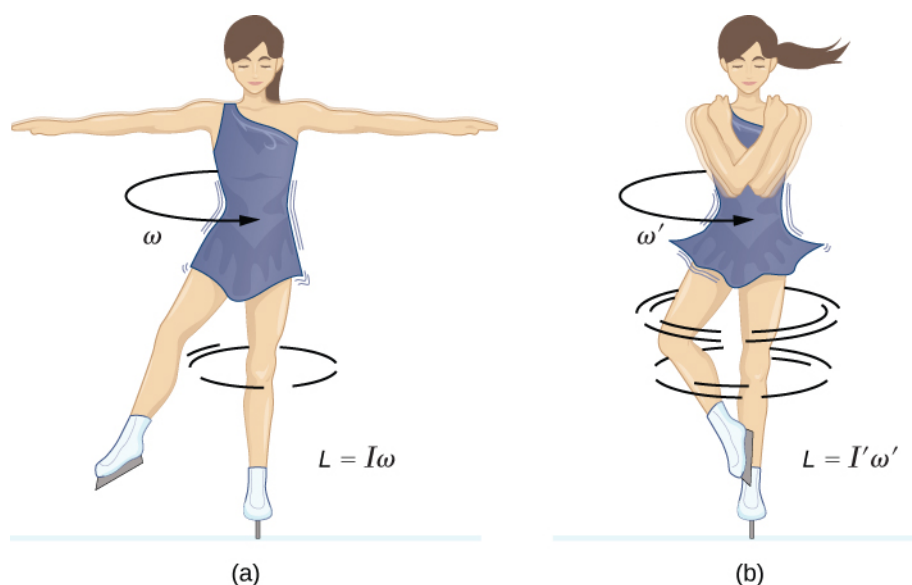


Figure 11.14 (a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. (b) Her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

It is interesting to see how the rotational kinetic energy of the skater changes when she pulls her arms in. Her initial rotational energy is

$$K_{\text{Rot}} = \frac{1}{2}I\omega^2,$$

whereas her final rotational energy is

$$K'_{\text{Rot}} = \frac{1}{2}I'(\omega')^2.$$

Since $I'\omega' = I\omega$, we can substitute for ω' and find

$$K'_{\text{Rot}} = \frac{1}{2}I'(\omega')^2 = \frac{1}{2}I'\left(\frac{I\omega}{I'}\right)^2 = \frac{1}{2}I\omega^2\left(\frac{I}{I'}\right) = K_{\text{Rot}}\left(\frac{I}{I'}\right).$$

Because her moment of inertia has decreased, $I' < I$, her final rotational kinetic energy has increased. The source of this additional rotational kinetic energy is the work required to pull her arms inward. Note that the skater's arms do not move in a perfect circle—they spiral inward. This work causes an increase in the rotational kinetic energy, while her angular momentum remains constant. Since she is in a frictionless environment, no energy escapes the system. Thus, if she were to extend her arms to their original positions, she would rotate at her original angular velocity and her kinetic energy would return to its original value.

The solar system is another example of how conservation of angular momentum works in our universe. Our solar system was born from a huge cloud of gas and dust that initially had rotational energy. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result of conservation of angular momentum (**Figure 11.15**).

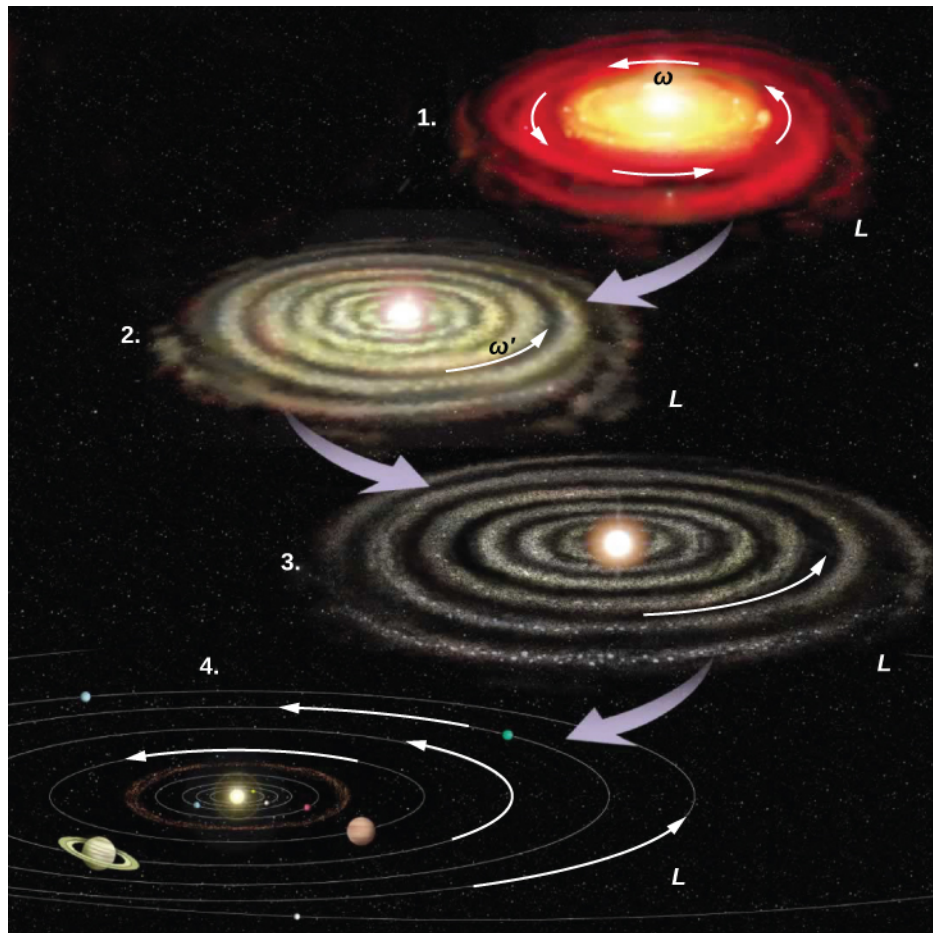


Figure 11.15 The solar system coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud. (credit: modification of work by NASA)

We continue our discussion with an example that has applications to engineering.

Example 11.7

Coupled Flywheels

A flywheel rotates without friction at an angular velocity $\omega_0 = 600 \text{ rev/min}$ on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it (**Figure 11.16**). Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together. (a) Use the law of conservation of angular momentum to determine the angular velocity ω of the combination. (b) What fraction of the initial kinetic energy is lost in the coupling of the flywheels?

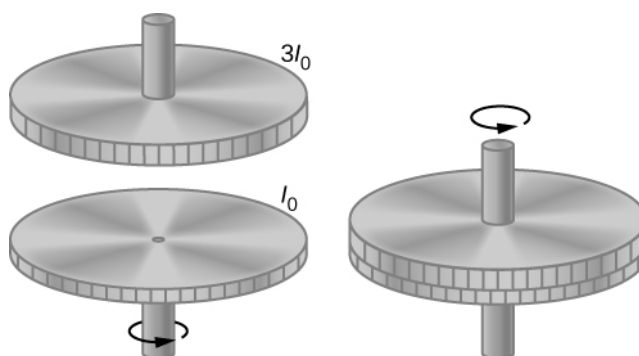


Figure 11.16 Two flywheels are coupled and rotate together.

Strategy

Part (a) is straightforward to solve for the angular velocity of the coupled system. We use the result of (a) to compare the initial and final kinetic energies of the system in part (b).

Solution

a. No external torques act on the system. The force due to friction produces an internal torque, which does not affect the angular momentum of the system. Therefore conservation of angular momentum gives

$$I_0 \omega_0 = (I_0 + 3I_0) \omega,$$

$$\omega = \frac{1}{4} \omega_0 = 150 \text{ rev/min} = 15.7 \text{ rad/s}.$$

b. Before contact, only one flywheel is rotating. The rotational kinetic energy of this flywheel is the initial rotational kinetic energy of the system, $\frac{1}{2} I_0 \omega_0^2$. The final kinetic energy is $\frac{1}{2} (4I_0) \omega^2 = \frac{1}{2} (4I_0) \left(\frac{\omega_0}{4} \right)^2 = \frac{1}{8} I_0 \omega_0^2$.

Therefore, the ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{\frac{1}{8} I_0 \omega_0^2}{\frac{1}{2} I_0 \omega_0^2} = \frac{1}{4}.$$

Thus, 3/4 of the initial kinetic energy is lost to the coupling of the two flywheels.

Significance

Since the rotational inertia of the system increased, the angular velocity decreased, as expected from the law of conservation of angular momentum. In this example, we see that the final kinetic energy of the system has decreased, as energy is lost to the coupling of the flywheels. Compare this to the example of the skater in **Figure 11.14** doing work to bring her arms inward and adding rotational kinetic energy.



11.4 Check Your Understanding A merry-go-round at a playground is rotating at 4.0 rev/min. Three children jump on and increase the moment of inertia of the merry-go-round/children rotating system by 25%. What is the new rotation rate?

Example 11.8

Dismount from a High Bar

An 80.0-kg gymnast dismounts from a high bar. He starts the dismount at full extension, then tucks to complete a number of revolutions before landing. His moment of inertia when fully extended can be approximated as a rod of length 1.8 m and when in the tuck a rod of half that length. If his rotation rate at full extension is 1.0 rev/s and he enters the tuck when his center of mass is at 3.0 m height moving horizontally to the floor, how many revolutions

can he execute if he comes out of the tuck at 1.8 m height? See **Figure 11.17**.

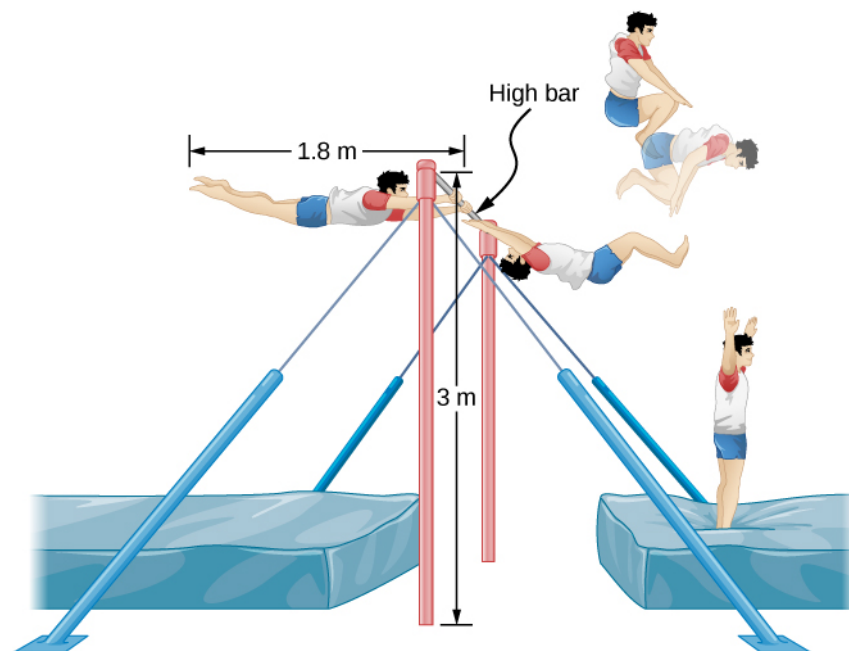


Figure 11.17 A gymnast dismounts from a high bar and executes a number of revolutions in the tucked position before landing upright.

Strategy

Using conservation of angular momentum, we can find his rotation rate when in the tuck. Using the equations of kinematics, we can find the time interval from a height of 3.0 m to 1.8 m. Since he is moving horizontally with respect to the ground, the equations of free fall simplify. This will allow the number of revolutions that can be executed to be calculated. Since we are using a ratio, we can keep the units as rev/s and don't need to convert to radians/s.

Solution

The moment of inertia at full extension is $I_0 = \frac{1}{12}mL^2 = \frac{1}{12}80.0 \text{ kg}(1.8 \text{ m})^2 = 21.6 \text{ kg} \cdot \text{m}^2$.

The moment of inertia in the tuck is $I_f = \frac{1}{12}mL_f^2 = \frac{1}{12}80.0 \text{ kg}(0.9 \text{ m})^2 = 5.4 \text{ kg} \cdot \text{m}^2$.

Conservation of angular momentum: $I_f \omega_f = I_0 \omega_0 \Rightarrow \omega_f = \frac{I_0 \omega_0}{I_f} = \frac{21.6 \text{ kg} \cdot \text{m}^2(1.0 \text{ rev/s})}{5.4 \text{ kg} \cdot \text{m}^2} = 4.0 \text{ rev/s}$.

Time interval in the tuck: $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(3.0 - 1.8)\text{m}}{9.8 \text{ m/s}^2}} = 0.5 \text{ s}$.

In 0.5 s, he will be able to execute two revolutions at 4.0 rev/s.

Significance

Note that the number of revolutions he can complete will depend on how long he is in the air. In the problem, he is exiting the high bar horizontally to the ground. He could also exit at an angle with respect to the ground, giving him more or less time in the air depending on the angle, positive or negative, with respect to the ground. Gymnasts must take this into account when they are executing their dismounts.

Example 11.9

Conservation of Angular Momentum of a Collision

A bullet of mass $m = 2.0 \text{ g}$ is moving horizontally with a speed of 500.0 m/s . The bullet strikes and becomes embedded in the edge of a solid disk of mass $M = 3.2 \text{ kg}$ and radius $R = 0.5 \text{ m}$. The cylinder is free to rotate around its axis and is initially at rest (**Figure 11.18**). What is the angular velocity of the disk immediately after the bullet is embedded?

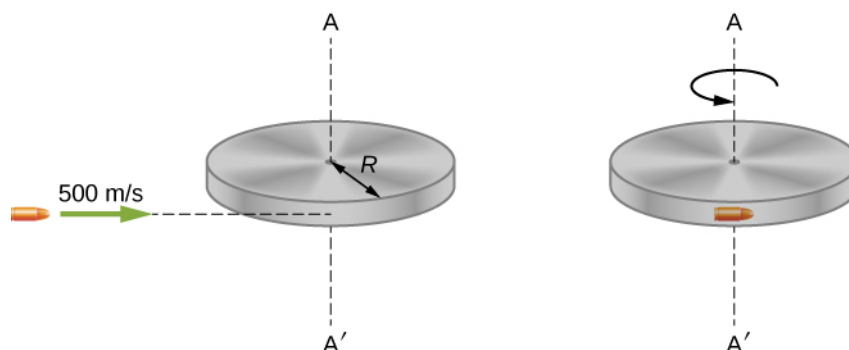


Figure 11.18 A bullet is fired horizontally and becomes embedded in the edge of a disk that is free to rotate about its vertical axis.

Strategy

For the system of the bullet and the cylinder, no external torque acts along the vertical axis through the center of the disk. Thus, the angular momentum along this axis is conserved. The initial angular momentum of the bullet is mvR , which is taken about the rotational axis of the disk the moment before the collision. The initial angular momentum of the cylinder is zero. Thus, the net angular momentum of the system is mvR . Since angular momentum is conserved, the initial angular momentum of the system is equal to the angular momentum of the bullet embedded in the disk immediately after impact.

Solution

The initial angular momentum of the system is

$$L_i = mvR.$$

The moment of inertia of the system with the bullet embedded in the disk is

$$I = mR^2 + \frac{1}{2}MR^2 = \left(m + \frac{M}{2}\right)R^2.$$

The final angular momentum of the system is

$$L_f = I\omega_f.$$

Thus, by conservation of angular momentum, $L_i = L_f$ and

$$mvR = \left(m + \frac{M}{2}\right)R^2\omega_f.$$

Solving for ω_f ,

$$\omega_f = \frac{mvR}{\left(m + \frac{M}{2}\right)R^2} = \frac{(2.0 \times 10^{-3} \text{ kg})(500.0 \text{ m/s})}{(2.0 \times 10^{-3} \text{ kg} + 1.6 \text{ kg})(0.50 \text{ m})} = 1.2 \text{ rad/s}.$$

Significance

The system is composed of both a point particle and a rigid body. Care must be taken when formulating the

angular momentum before and after the collision. Just before impact the angular momentum of the bullet is taken about the rotational axis of the disk.

11.4 | Precession of a Gyroscope

Learning Objectives

By the end of this section, you will be able to:

- Describe the physical processes underlying the phenomenon of precession
- Calculate the precessional angular velocity of a gyroscope

Figure 11.19 shows a gyroscope, defined as a spinning disk in which the axis of rotation is free to assume any orientation. When spinning, the orientation of the spin axis is unaffected by the orientation of the body that encloses it. The body or vehicle enclosing the gyroscope can be moved from place to place and the orientation of the spin axis will remain the same. This makes gyroscopes very useful in navigation, especially where magnetic compasses can't be used, such as in manned and unmanned spacecraft, intercontinental ballistic missiles, unmanned aerial vehicles, and satellites like the Hubble Space Telescope.



Figure 11.19 A gyroscope consists of a spinning disk about an axis that is free to assume any orientation.

We illustrate the **precession** of a gyroscope with an example of a top in the next two figures. If the top is placed on a flat surface near the surface of Earth at an angle to the vertical and is not spinning, it will fall over, due to the force of gravity producing a torque acting on its center of mass. This is shown in **Figure 11.20(a)**. However, if the top is spinning on its axis, rather than topple over due to this torque, it precesses about the vertical, shown in part (b) of the figure. This is due to the torque on the center of mass, which provides the change in angular momentum.